ORDINARY AND $\mathbb{Z}_2$-GRADED COCHARACTERS OF $UT_2(E)$

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Abstract. Let E be the infinite dimensional Grassmann algebra over a field F of characteristic 0. In this paper we compute the ordinary and the $\mathbb{Z}_2$-graded cocharacters of the algebra of 2×2 upper triangular matrices with coefficients in E, using the tool of proper Hilbert series.

1. Introduction

Varieties of associative algebras over a field F are in bijective correspondence with the ideals of the free associative algebra $F\langle X\rangle = F\langle x_1, x_2, \ldots\rangle$ that is invariant under all the endomorphisms of $F\langle X\rangle$. Such ideals are called T-ideals and they are the ideals of the polynomial identities satisfied by any algebra of the variety. For the study of the T-ideals over a field of characteristic zero, a fundamental tool is given by the representation theory of the linear and symmetric groups. Moreover, the theorems of Kemer about the classification of the T-ideals of $F\langle X\rangle$ show that the notion of grading of an algebra defined by a group is another key ingredient for such study. In particular, one has that any proper T-ideal of $F\langle X\rangle$ is the ideal of the polynomial identities satisfied by the Grassmann envelope of a suitable $\mathbb{Z}_2$-graded algebra (also called superalgebra) of finite dimension. The work of Giambruno and Zaicev [8] has contributed to clarify why the notion of PI-exponent is crucial for a classification of the T-ideals in terms of growth of the sequence of their codimensions. Recall that the n-th codimension of a T-ideal is defined as the degree of the representation of the group S_n on the vector space of the multilinear polynomials of $F\langle X\rangle$ of degree n modulo the considered T-ideal. In [8] the authors prove that the minimal varieties with respect to a fixed exponent are determined by the T-ideals of the Grassmann envelope of the so called minimal superalgebras. Over an algebraic closed field, such superalgebras can be realized as graded subalgebras of block-triangular matrix algebras equipped with a suitable $\mathbb{Z}_2$-grading. Precisely, the blocks along the main diagonal are simple superalgebras of finite dimension. Then, by the Theorem of Lewin [10] one has that the T-ideals of the identities satisfied by the minimal superalgebras and their Grassmann envelopes are products of the T-ideals corresponding to the diagonal blocks. Such results allow hence to solve in the positive a conjecture due to Drensky [5, 6] about the factorability of the T-ideals of minimal varieties as a product of verbally prime T-ideals. Moreover, Berele and Regev [1] proved a formula that relates the sequence of ordinary cocharacters of a product of T-ideals to the sequences of cocharacters of these ideals. Recently, in [3] Di Vincenzo and La Scala introduce the notion of G-regularity of an algebra, if G is a finite abelian group, and they prove that $T_G(R) = T_G(A)T_G(B)$, provided that at least one of the algebras A, B is G-regular. They also proved for suitable groups G and for A, B that the G-regularity of A or B is a necessary condition for the ideal $T_G(R)$ to be factorable. They proved a formula allowing us to compute the sequence of graded cocharacters of a superalgebra R such that $T_2(R) = T_2(A)T_2(B)$ starting from the corresponding sequences of A and B.

The varieties with exponent two were characterized by Giambruno and Zaicev in [9]. They listed five algebras such that a variety has exponent less then or equal to 2 if and only if it does not contain any of E and $UT_2(F)$. These algebras generate all the possible minimal varieties of exponent strictly greater

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than two. One of these algebras is $G = \begin{pmatrix} E & E \\ 0 & E^0 \end{pmatrix}$, where $E = E^0 + E^1$ is the natural $\mathbb{Z}_2$-grading of E. In this paper, we compute firstly the Hilbert Series (and so the cocharacters) of G, then intends to contribute to this line of research by studying the ordinary and the $\mathbb{Z}_2$-graded cocharacters of

$$UT_2(E) = \begin{pmatrix} E & E \\ 0 & E \end{pmatrix},$$

that is a minimal algebra respect to its exponent, using the fundamental tool of proper Hilbert series.

2. Ordinary and $\mathbb{Z}_2$-graded structure

All fields in this paper are assumed to be of characteristic 0.

Let F be a field and A be an associative F-algebra. Let now $X = \{x_1, x_2, \ldots\}$ be a countable set of variables. We denote by $F\langle X\rangle$ the free associative algebra generated by X and by $T(A)$ the intersection of the kernels of all homomorphisms $F\langle X\rangle \to A$. Then $T(A)$ is a two-sided ideal of $F\langle X\rangle$ and its elements are called polynomial identities of the algebra A. If $T(A)$ is not trivial, A is said to be a *polynomial identity algebra* (P.I. algebra). Note that $T(A)$ is stable under the action of any endomorphism of the algebra $F\langle X\rangle$. Any ideal of $F\langle X\rangle$ which verifies such property is said to be a T-ideal. Clearly, any T-ideal I is the ideal of the polynomial identities of the algebra $F\langle X\rangle/I$. Note also that for a P.I. algebra A, the quotient algebra $F\langle X\rangle/T(A)$ is the relatively free algebra for the variety of algebras generated by A.

Definition 2.1. *For $n \in \mathbb{N}$, the vector space*

$$V_n := \mathrm{span}_F \left\langle x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} | \sigma \in S_n,\ x_i \in X \right\rangle$$

is called the space of multilinear polynomials of degree n.

Since the characteristic of the ground field F is zero, a standard process of multilinearization shows that $T(A)$ is generated, as a T-ideal, by the subspaces $V_n \cap T(A)$. Actually, it is more efficient to study the factor space

$$V_n(A) = V_n/(V_n \cap T(A)).$$

An effective tool to this end is provided by the representation theory of the symmetric group. Indeed, one can notice that V_n is an S_n-module with respect to the natural left action, and $V_n \cap T(A)$ is an S_n-submodule, hence the factor space $V_n(A)$ is an S_n-module, too. We shall denote by $\chi_n(A)$ its character, called the *n-th cocharacter of A*. For a more detailed account about the representation theory of symmetric group, we remand to [12].

The *commutator* of $a, b \in A$ is the Lie product $[a, b] := ab - ba$. One defines inductively higher (left-normed) commutators by setting $[a_1, \cdots, a_n] := [[a_1, \ldots, a_{n-1}], a_n]$, for any $n \geq 2$. By the Poincarè-Birkhoff-Witt theorem, $F\langle X\rangle$ has a basis $\{x_1^{s_1} \cdots x_r^{s_r} u_1^{m_1} \cdots u_n^{m_n} | s_i, m_j \geq 0,\ r, n \in \mathbb{N}\}$, where $u_1, u_2, \ldots$ are higher commutators. We denote by $B(X)$ the unitary subalgebra of $F\langle X\rangle$ generated by commutators, called the algebra of *proper polynomials.* It is well known that $B(X) \cap T(A)$ generates the whole $T(A)$ as a T-ideal. Let us denote $B(A) := B(X)/B(X)\cap T(A)$. We shall denote Γ_n the set of multilinear polynomials of V_n which are proper. It is not difficult to see that Γ_n is a left S_n-submodule of V_n and the same holds for $\Gamma_n \cap T(A)$. Hence the factor module

$$\Gamma_n(A) = \Gamma_n/(\Gamma_n \cap T(A))$$

is an S_n-submodule of $V_n(A)$. We shall denote by ξ_n its character (*n-th proper cocharacter of A*). For a more detailed account about proper cocharacters we refer to the book of Drensky ([7], Chapters 4, 12).

The following result of Drensky relates the ordinary cocharacters of a P.I. algebra A with the proper cocharacters of A:

Proposition 2.1. *(Drensky, [7], Theorem 12.5.4) Let A be a P.I. algebra and let $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)\chi_\lambda$ its n-th cocharacter. Let $\xi_p(A) = \sum_{\nu \vdash p} k_\nu(A)\chi_\nu$ its p-th proper cocharacter, then*

$$m_\lambda(A) = \sum_{\nu \in S} k_\nu(A),$$

where $S = \{\nu = (\nu_1, \dots, \nu_n) \mid \lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \cdots \geq \lambda_n \geq \nu_n\}$.

We say that A is a $\mathbb{Z}_2$-graded algebra if $A = A^0 \oplus A^1$, where $A^0, A^1 \subseteq A$ are subspaces and $A_g A_h \subseteq A_{g+h}$ holds for $g, h \in \mathbb{Z}_2$. The subspace A_g is called the homogeneous component of A of degree g. We say that the elements $a \in A_g$ are homogeneous of degree g and we denote their degrees as $|a| = g$. One defines $\mathbb{Z}_2$-graded: subspaces of A, A-modules, homomorphisms and so on, in a standard way, see for example [1].

We denote by $F\langle X|\mathbb{Z}_2\rangle$ the free associative algebra generated by $X = Y \cup Z$, where $Y = \{y_1, y_2, \dots\}$ and $Z = \{z_1, z_2, \dots\}$ are two countable sets of disjoint variables. Given a map $|\ | : X \to \mathbb{Z}_2$, we can define a $\mathbb{Z}_2$-grading on $F\langle X|\mathbb{Z}_2\rangle$ if we set $|w| = |x_{j_1}| + \cdots + |x_{j_n}|$ for any monomial $w = x_{j_1} \cdots x_{j_n} \in F\langle X|\mathbb{Z}_2\rangle$. Then, the homogeneous component $F\langle X|\mathbb{Z}_2\rangle_g \subseteq F\langle X|\mathbb{Z}_2\rangle$ is the subspace spanned by all monomials of degree g. Because $\mathbb{Z}_2$ is a finite group, we assume that the fibers of the map $|\ |$ are all infinite. If A is a $\mathbb{Z}_2$-graded algebra, we denote by $T_{\mathbb{Z}_2}(A)$ the intersection of the kernels of all $\mathbb{Z}_2$-graded homomorphisms $F\langle X|\mathbb{Z}_2\rangle \to A$. Then $T_{\mathbb{Z}_2}(A)$ is a graded two-sided ideal of $F\langle X|\mathbb{Z}_2\rangle$ and its elements are called $\mathbb{Z}_2$-graded polynomial identities of the algebra A. Note that $T_{\mathbb{Z}_2}(A)$ is stable under the action of any $\mathbb{Z}_2$-graded endomorphism of the algebra $F\langle X|\mathbb{Z}_2\rangle$. Any $\mathbb{Z}_2$-graded ideal of $F\langle X|\mathbb{Z}_2\rangle$ which verifies such property is said to be a $T_{\mathbb{Z}_2}$-ideal. Clearly, any $T_{\mathbb{Z}_2}$-ideal I is the ideal of the $\mathbb{Z}_2$-graded polynomial identities of the graded algebra $F\langle X|\mathbb{Z}_2\rangle/I$. Note also that for a $\mathbb{Z}_2$-graded algebra A, the quotient algebra $F\langle X|\mathbb{Z}_2\rangle/T_{\mathbb{Z}_2}(A)$ is the relatively free algebra for the variety of graded algebras generated by A.

Definition 2.2. *For $n \in \mathbb{N}$, the vector space*

$$V_n^{\mathbb{Z}_2} := \operatorname{span}_F \left\langle x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} | \sigma \in S_n,\ x_i = y_i \text{ or } x_i = z_i \right\rangle$$

is called the space of $\mathbb{Z}_2$-graded multilinear polynomials of degree n.

Since the charcteristic of the ground field F is zero, a standard process of multilinearization shows that $T_{\mathbb{Z}_2}(A)$ is generated, as a $T_{\mathbb{Z}_2}$-ideal, by the subspaces $V_n^{\mathbb{Z}_2} \cap T_{\mathbb{Z}_2}(A)$. Actually, it is more efficient to study the factor space

$$V_n^{\mathbb{Z}_2}(A) = V_n^{\mathbb{Z}_2}/(V_n^{\mathbb{Z}_2} \cap T_{\mathbb{Z}_2}(A)).$$

One can notice again that $V_n^{\mathbb{Z}_2}$ is an S_n-module with respect to the natural left action, and $V_n^{\mathbb{Z}_2} \cap T_{\mathbb{Z}_2}(A)$ is an S_n-submodule, hence the factor space $V_n^{\mathbb{Z}_2}(A)$ is an S_n-module, too. We shall denote by $\chi_n^{\mathbb{Z}_2}(A)$ its character, called the *n-th $\mathbb{Z}_2$-graded cocharacter of A*.

The study of the structure of $V_n^{\mathbb{Z}_2}(A)$ can be simplified by considering "smaller" spaces of multilinear polynomials. To be more precise, for fixed $l,\ m$, set

$$V_{l,m} := \operatorname{span}_F \left\langle w \text{ monomials of } V_{l+m}^{\mathbb{Z}_2} | y_1, \dots, y_l, z_{l+1}, \dots, z_{l+m} \text{ occur in } w \right\rangle.$$

Setting $n := l + m$, and $S_l \times S_m = Sym(\{1, \dots, l\}) \times Sym(\{l+1, \dots, l+m\}) \leq S_n$, the space $V_{l,m}$ is an $S_l \times S_m$-module, and the subspace $V_{l,m} \cap T_{\mathbb{Z}_2}(A)$ is a submodule. Therefore one can form the factor

$S_l \times S_m$-module

$$V_{l,m}(A) := V_{l,m}/(V_{l,m} \cap T_{\mathbb{Z}_2}(A)).$$

We shall denote by $\chi_{l,m}(A)$ its $S_l \times S_m$-character.

Now we give a small account on the representation theory of the groups $S_l \times S_m$ $(l + m = n)$. The irreducible $S_l \times S_m$-characters are in "one to one" correspondence with the pairs of partitions (λ, μ) of l and m respectively; in this case we write $\lambda \vdash l$, $\mu \vdash m$, and $|\lambda| = l$, $|\mu| = m$. More precisely, if χ_ν denotes the irreducible $S_{|\nu|}$-character associated to the partition ν, then the irreducible $S_l \times S_m$-character associated to (λ, μ) is $\chi_{\lambda,\mu} = \chi_\lambda \otimes \chi_\mu$.

3. Hilbert series and proper Hilbert series of P.I. algebras

In this section we talk about a tool used in the study of algebras in general, the Hilbert series. In particular, we emphasize the relationship between the Hilbert series and the combinatorial method of the Littlewood-Richardson rule.

We say that the vector space V is $\mathbb{Z}_m$-graded if $V = \bigoplus_{(n_1,\ldots,n_m)\in\mathbb{Z}_m} V^{(n_1,\ldots,n_m)}$, and $V^{(n_1,\ldots,n_m)}$ is a vector subspace of V.

Definition 3.1. *Let $V = \sum_{n\in\mathbb{Z}} V^{(n_1,\ldots,n_m)}$ be a $\mathbb{Z}^m$-graded vector space and let $\dim_F V^{(n_1,\ldots,n_m)} < \infty$. The formal power series*

$$H(V, t_1, \ldots, t_m) = \sum_n \dim_F V^{(n_1,\ldots,n_m)} t_1^{n_1} \cdots t_m^{n_m}$$

is called the Hilbert Series of V in the variables $t_1, \ldots, t_m$.

Let A be a P.I. algebra over F. It is well known that $T(A)$ is a multi-homogeneous ideal of $F\langle X\rangle$. Then, if $\bar{t} := (t_1, \ldots, t_m)$, we denote by

$$H(A, \bar{t}) := H\left(F\langle x_1, x_2, \ldots, x_m\rangle/(F\langle x_1, x_2, \ldots, x_m\rangle \cap T(A)), \bar{t}\right)$$

the Hilbert series of the relatively free algebra in m variables.

Hilbert series is related with usual operations between graded vector spaces. In fact

Proposition 3.1. *Let V, W be $\mathbb{Z}_m$-graded vector spaces and U be a $\mathbb{Z}_m$-graded vector subspace of V. Then*

- $H(V \oplus W, t_1, \ldots, t_m) = H(V, t_1, \ldots, t_m) + H(W, t_1, \ldots, t_m)$
- $H(V \otimes W, t_1, \ldots, t_m) = H(V, t_1, \ldots, t_m) \cdot H(W, t_1, \ldots, t_m)$
- $H(V, t_1, \ldots, t_m) = H(V/U, t_1, \ldots, t_m) + H(U, t_1, \ldots, t_m)$.

Definition 3.2. *Let T be a tableau of shape λ filled in with natural numbers $\{1, \ldots, k\}$ and let d_i be the multiplicity of i in T. Let*

$$S_\lambda = \sum_{T_\lambda \text{ semistandard}} t^{T_\lambda},$$

where $t^{T_\lambda} = t_1^{d_1} t_2^{d_2} \cdots t_k^{d_k}$. We say S_λ is the Schur Function of λ in the variables $t_1, \ldots, t_k$.

It is well known (see [12], Chapter 4) that the product of the Schur's functions

$$S_\lambda(t_1, \ldots, t_k) S_\mu(t_1, \ldots, t_k)$$

corresponds in a natural way to

$$(\lambda \otimes \mu)^{\uparrow S_n},$$

where $\lambda \vdash l$, $\mu \vdash m$ and $l + m = n$. Then in the computation of products of Schur functions we are allowed to use the combinatorial tool of the Littlewwod-Richardson Rule. For a more detailed account about the Littlewood-Richardson rule, see [12].

Hilbert series is strictly connected to the sequence of cocharacters of P.I. algebras. Indeed, we have the following proposition:

Proposition 3.2. *Let A be a P.I. algebra and let $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)\chi_\lambda$ its n-th cocharacter. Let $H(A, t_1, \ldots, t_k)$ the Hilbert series of A, then*

$$H(A, t_1, \ldots, t_k) = \sum_{n \geq 0} \sum_{\lambda \in H(k,0,n)} m_\lambda(A) S_\lambda(t_1, \ldots, t_k).$$

At the light of the previous proposition, in order to compute the cocharacters of a P.I. algebra A, it is sufficient to argue in terms of its Hilbert series.

We can define in an analogous way the so called *proper Hilbert series* of a P.I. algebra.

Definition 3.3. *Let A be a P.I. algebra over F, then, if $\bar{t} := (t_1, \ldots, t_m)$, we denote by*

$$H^B(A, \bar{t}) := H\left(B(x_1, \ldots, x_m)/(B(x_1, \ldots, x_m) \cap T(A), \bar{t})\right)$$

the Hilbert series of the relatively free algebra of proper polynomial in m variables.

Hilbert series and proper Hilbert series are related by the following proposition (see for example [7], Theorem 4.3.12 (i)):

Proposition 3.3. *Let A be a P.I. algebra. Then*

$$H(A, \bar{t}) = \prod_{i=1}^{m} \frac{1}{1 - t_i} \cdot H^B(A, \bar{t}).$$

Remark 3.1. *The rational function $\prod_{i=1}^m \frac{1}{1-t_i}$ can be expressed in terms of Schur functions as $\sum_{k \geq 0} S_{(k)}$.*

4. The Theorem of Lewin and Hilbert series of upper triangular matrices

Let I and J be two T-ideals. Consider the quotient algebras $F\langle X\rangle/I, F\langle X\rangle/J$ and let U be a $F\langle X\rangle/I$-$F\langle X\rangle/J$-bimodule. We define:

$$R = \begin{pmatrix} F\langle X\rangle/I & U \\ 0 & F\langle X\rangle/J \end{pmatrix}.$$

Fix u_i a countable set of elements of U. Then $\varphi : x_i \to a_i$ defines an algebra homomorphism, where:

$$a_i = \begin{pmatrix} x_i + I & u_i \\ 0 & x_i + J \end{pmatrix}.$$

If $f(x_1, \ldots, x_n) \in F\langle X\rangle$ one has that $f(x_1, \ldots, x_n) \to f(a_1, \ldots, a_n)$, where:

$$f(a_1, \ldots, a_n) = \begin{pmatrix} f(x_1, \ldots, x_n) + I & \delta(f) \\ 0 & f(x_1, \ldots, x_n) + J \end{pmatrix}$$

and $\delta(f)$ is some element of U. Then $IJ \subseteq ker(\varphi) = I \cap J \cap ker(\delta(f))$ and $\delta : F\langle X\rangle \to U$ is an F-derivation.

Theorem 4.1. *(Lewin [10]). If $\{u_i\}$ is a countable free set of elements of the bimodule U then for the homomorphism φ defined by $\{u_i\}$, we have* $\ker(\varphi) = IJ$.

Corollary 4.1. *If the bimodule U contains a countable free set $\{u_i\}$ for any i, then $T(R) = IJ$.*

Proof. We have to prove only the "left to right" inclusion. By the Theorem of Lewin $\ker(\varphi) = IJ$ but $T(R) \subseteq \ker(\varphi) = IJ$ and we are done. $\square$

Let A, B be F-algebras, suppose they are both P.I. and let U be an $A-B$-bimodule, then we can consider

$$R = \begin{pmatrix} A & U \\ 0 & B \end{pmatrix}$$

that is still an F-algebra and a P.I. algebra such that $T(R) \supseteq T(A)T(B)$. Suppose now that $T(R) = T(A)T(B)$, then we have a formula that relates the Hilbert series of R with the Hilbert series of A and B.

$$H(R,\bar{t}) = H(A,\bar{t}) + H(B,\bar{t}) + \left(S_{(1)} - 1\right) H(A,\bar{t})H(B,\bar{t}). \tag{4.1}$$

The equation (4.1) gives us the possibility to compute the proper Hilbert series of the algebra R using Proposition 3.3:

$$H^B(R,\bar{t}) = H^B(A,\bar{t}) + H^B(B,\bar{t}) + \frac{\left(S_{(1)} - 1\right)}{\prod_{i=1}^{m}(1-x_i)} H^B(A,\bar{t})H^B(B,\bar{t}). \tag{4.2}$$

5. Cocharacters of G

In this section, we compute the Hilbert series of G that is one of the five minimal algebra of exponent 2 in the classification of Giambruno and Zaicev [9]. This variety has been studied by Stoyanova-Venkova in [13], too.

The matrix algebra

$$G = \begin{pmatrix} E & E \\ 0 & E^0 \end{pmatrix}$$

is a P.I. algebra such that $T(G) = T(E)T(E^0)$. We will use the equation (4.1) to compute its Hilbert series, then we can compute its cocharacter sequence at the light of Proposition 3.2.

Let us divide the proof in some lemmas using the well known facts that $H(E,\bar{t}) = \sum S_{(k,1^l)}$ (see, [11]) and $H(E^0,\bar{t}) = \sum S_{(k)}$.

Lemma 5.1. $H(E,\bar{t})H(E^0,\bar{t}) = \sum m_\lambda S_\lambda$, *where* $\lambda = (k_1, k_2, 1^l)$ *and*

$$m_\lambda = 2(k_1 - k_2 + 1) \text{ if } k_2 \geq 1$$
$$m_\lambda = k_1 + 1 \text{ if } k_2 = l = 0.$$

Proof. Suppose $l \geq 1$, then by the Littlewood-Richardson rule one easily has that $(k_1, k_2, 1^l)$ comes from the tensor product $(k_1 - i, 1^{l+1}) \otimes (i + k_2 - 1)$ or from $(k_1 - i, 1^l) \otimes (i + k_2)$ for $i = 0, \ldots, k_1 - k_2$, so the total multiplicity is $2(k_1 - k_2 + 1)$. We argue analogously for the other case.

$\square$

Lemma 5.2. $S_{(1)}H(E,\bar{t})H(E^0,\bar{t}) = \sum m_\lambda S_\lambda$, *where* $\lambda = (k_1, k_2, 1^l)$ *or* $\lambda = (k_1, k_2, 2, 1^l)$. *If* $\lambda = (k_1, k_2, 1^l)$, *then*

$$m_\lambda = 6(k_1 - k_2 + 1) \text{ if } k_2 \geq 2 \text{ and } l \geq 1$$
$$m_\lambda = 4k_1 - 2 \text{ if } k_2 = 0 \text{ and } l \geq 2$$
$$m_\lambda = 3k_1 - 1 \text{ if } k_2 = 0 \text{ and } l = 1$$
$$m_\lambda = 4(k_1 - k_2 + 1) \text{ if } l = 0$$
$$m_\lambda = k_1 \text{ if } k_2 = l = 0.$$

If $\lambda = (k_1, k_2, 2, 1^l)$, *then*

$$m_\lambda = 2(k_1 - k_2 + 1).$$

Proof. Suppose $\lambda = (k_1, k_2, 1^l)$ and $l \geq 1$, then by the Littlewood-Richardson rule one easily has that $(k_1, k_2, 1^l)$ comes from the tensor product $(k_1, k_2, 1^{l-1}) \otimes (1)$, or $(k_1, k_2 - 1, 1^l) \otimes (1)$ or $(k_1 - 1, k_2, 1^l) \otimes (1)$. By previous Lemma, their multiplicities are respectively equal to $2(k_1 - k_2 + 1)$, $2(k_1 - k_2 + 2)$ and $2(k_1 - k_2)$, so the total multiplicity equals $2(k_1 - k_2 + 1) + 2(k_1 - k_2 + 2) + 2(k_1 - k_2) = 6(k_1 - k_2 + 1)$.

Suppose now $\lambda = (k_1, k_2)$, then one has that (k_1, k_2) comes from the tensor product $(k_1, k_2 - 1) \otimes (1)$, or $(k_1 - 1, k_2) \otimes (1)$. By the previous Lemma, their multiplicities are respectively equal to $2(k_1 - k_2 + 2)$ and $2(k_1 - k_2)$, so the total multiplicity equals $2(k_1 - k_2 + 2) + 2(k_1 - k_2) = 4(k_1 - k_2 + 1)$.

Finally, if $\lambda = (k_1, k_2, 2, 1^l)$ one has that $(k_1, k_2, 2, 1^l)$ comes from the tensor product $(k_1, k_2, 1^{l+1}) \otimes (1)$ only. By previous Lemma, its multiplicity equals $2(k_1 - k_2 + 1)$ and we are done. The other cases are treated similarly. □

Lemma 5.3. $(S_{(1)} - 1)H(E, \bar{t})H(E^0, \bar{t}) = \sum m_\lambda S_\lambda$, *where* $\lambda = (k_1, k_2, 1^l)$ *or* $\lambda = (k_1, k_2, 2, 1^l)$. *If* $\lambda = (k_1, k_2, 1^l)$, *then*

$$\begin{gathered} m_\lambda = 4(k_1 - k_2 + 1) \textit{if} \;\; k_2 \geq 2 \textit{ and } l \geq 1 \\ m_\lambda = 2k_1 - 2 \textit{ if } k_2 = 0 \textit{ and } l \geq 2 \\ m_\lambda = k_1 - 1 \textit{ if } k_2 = 0 \textit{ and } l = 1 \\ m_\lambda = 2(k_1 - k_2 + 1) \textit{ if } l = 0 \\ m_\lambda = -1 \textit{ if } k_2 = l = 0. \end{gathered}$$

If $\lambda = (k_1, k_2, 2, 1^l)$, *then*

$$m_\lambda = 2(k_1 - k_2 + 1).$$

Proof. We have just to use Lemmas 5.2 and 5.3. □

Finally, the complete Hilbert series.

Proposition 5.1. $H(G, \bar{t}) = \sum m_\lambda S_\lambda$, *where* $\lambda = (k_1, k_2, 1^l)$ *or* $\lambda = (k_1, k_2, 2, 1^l)$. *If* $\lambda = (k_1, k_2, 1^l)$, *then*

$$\begin{gathered} m_\lambda = 4(k_1 - k_2 + 1) \textit{ if } \;\; k_2 \geq 2 \textit{ and } l \geq 1 \\ m_\lambda = 2k_1 - 1 \textit{ if } k_2 = 0 \textit{ and } l \geq 2 \\ m_\lambda = 2(k_1 - k_2 + 1) \textit{ if } l = 0 \\ m_\lambda = k_1 \textit{ if } l = 1 \\ m_\lambda = 1 \textit{ if } k_2 = l = 0. \end{gathered}$$

If $\lambda = (k_1, k_2, 2, 1^l)$, *then*

$$m_\lambda = 2(k_1 - k_2 + 1).$$

Proof. Straightforward. □

Example 5.1. *Using Proposition* 5.1, *we have that*

$$\chi_1(G) = (1),$$

$$\chi_2(G) = (2) + (1^2),$$

$$\chi_3(G) = (3) + 2(2, 1) + (1^3),$$

$$\chi_4(G) = (4) + 3(3, 1) + 2(2^2) + 3(2, 1^2) + (1^4),$$

$$\chi_5(G) = (5) + 4(4, 1) + 4(3, 2) + 5(3, 1^2) + 4(2^2, 1) + 3(2, 1^3) + (1^5),$$

$$\chi_6(G) = (6) + 5(5, 1) + 6(4, 2) + 7(4, 1^2) + 8(3, 2, 1) + 2(3^2) + 5(3, 1^3) + 2(2^3) + 4(2^2, 1^2) + 3(2, 1^4) + (1^6).$$

6. Cocharacters of $UT_2(E)$

We compute the Hilbert series of $UT_2(E)$ starting from its proper Hilbert series and using the combinatorial properties of the Littlewood-Richardson rule.

Consider the matrix algebra

$$UT_2(E) = \begin{pmatrix} E & E \\ 0 & E \end{pmatrix}$$

and let $R := UT_2(E)$. At the light of Theorem 4.1 and its following Corollary, we have that $T(R) = T(E)T(E)$, then the Proposition (3.3) gives us that

$$H^B(R, \bar{t}) = 2H^B(E, \bar{t}) + \frac{(S_{(1)} - 1)}{\prod_{i=1}^{m}(1 - x_i)}(H^B(E, \bar{t}))^2. \tag{6.1}$$

It is convenient to break the computation of $H^B(R, \bar{t})$ into some lemmas.

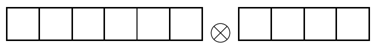
Lemma 6.1. *$(H^B(E, \bar{t}))^2 = \sum m_\lambda S_\lambda$, where*

$$\lambda = (2^{\mu_2}, 1^{\mu_1})$$

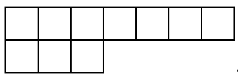
and

$$m_\lambda = \begin{cases} \frac{\mu_1 - \mu_2}{2} + 1 & \text{if } \mu_2 \text{ even} \\ \frac{\mu_1 - \mu_2}{2} & \text{if } \mu_2 \text{ odd.} \end{cases}$$

Proof. It is well known that $H^B(E, \bar{t}) = \sum_{k \geq 0} S_{(1^{2k})}$ (see [7], Chapters 4, 12), so let us check out the multiplicities of $\left[(\sum_{k \geq 0} S_{(2k)})^2\right]'$. By Littlewood-Richardson rule we have that the only allowed partitions in the tensor product



are of the type $\mu = (\mu_1, \mu_2)$ as in the picture below

$\mu =$  .

Due to the parity of $|\mu|$, we have that μ_1, μ_2 are both even or both odd numbers. If μ_2 is even we have exactly $\frac{\mu_1 - \mu_2}{2} + 1$ allowed partitions. If μ_2 is odd, μ doesn't occur as $(\mu_1) \otimes (\mu_2)$ so we have exactly $\frac{\mu_1 - \mu_2}{2}$ allowed partitions. □

Remark 6.1. *The partitions $\lambda_1 = (2, 1^l)$, if l is odd and $\lambda_2 = (2)$ are not allowed in the previous decomposition. In fact, $|\lambda_1|$ is odd and λ_2 comes only from $1 \otimes (2)$ or $(1) \otimes (1)$ and both of (1) and (2) do not appear in the decomposition of $H^B(E, \bar{t})$.*

Lemma 6.2. *$\frac{1}{\prod_{i=1}^{m}(1 - x_i)}(H^B(E, \bar{t}))^2 = \sum m_\lambda S_\lambda$, where*

$$\lambda = (k, 2^m, 1^l)$$

and

$$m_\lambda = \begin{cases} \frac{l}{2} + 1 & \text{if } l \text{ is even} \\ \frac{l-1}{2} + 1 & \text{if } l \text{ is odd} \end{cases} \quad \text{if } k=m=0.$$

$$m_\lambda = l + 1 \text{ otherwise.}$$

Proof. We argue only for $\lambda=(k,2^m,1^l)$, with $m\geq 1$, because the case $m=0$ is treated similarly. By the Littlewood-Richardson rule one has that $(k,2^m,1^l)$ occurs in $(2^{m+1},1^l)\otimes(k-2)$, $(2^m,1^l)\otimes(k)$ if l is even, $(2^{m+1},1^{l-1})\otimes(k-1)$, $(2^m,1^{l+1})\otimes(k-1)$ if l is odd, then by Lemma 6.1, $(2^{m+1},1^l)$, $(2^m,1^l)$, $(2^{m+1},1^{l-1})$ and $(2^m,1^{l+1})\otimes(k-1)$, have multiplicities respectively equal to $\frac{l}{2}$, $\frac{l}{2}+1$, $\frac{l-1}{2}$, and $\frac{l+1}{2}+1$ if m is even, equal to $\frac{l}{2}+1$, $\frac{l}{2}$, $\frac{l-1}{2}+1$, and $\frac{l+1}{2}$ if m is odd. Then $m_\lambda=\frac{l}{2}+\frac{l}{2}+1=l+1$ if l is even and m is even; $m_\lambda=\frac{l}{2}+1+\frac{l}{2}=l+1$ if l is even and m is odd; $m_\lambda=\frac{l-1}{2}+\frac{l+1}{2}+1=l+1$ if l is odd and m is even; $m_\lambda=\frac{l-1}{2}+1+\frac{l+1}{2}=l+1$ if l is odd and m is odd, from which the assertion.

If $\lambda=(1^l)$, By the Littlewood-Richardson rule one has that (1^l) occurs in $(1^l)\otimes 1$ if l is even, $(1^{l-1})\otimes(1)$ if l is odd, then by Lemma 6.1, it has multiplicities respectively equal to $\frac{l}{2}+1$ and $\frac{l-1}{2}+1$ and we are done. □

Lemma 6.3. $\frac{S_{(1)}}{\prod_{i=1}^m(1-x_i)}(H^B(E,\bar t))^2=\sum m_\lambda S_\lambda$, *where*

$$\lambda=(k,2^m,1^l)$$
$$\textit{or } \lambda=(k,3,2^m,1^l).$$

If $\lambda=(k,2^m,1^l)$, *then*

$$m_\lambda=3(l+1) \textit{ if } \; k\geq 3, m\geq 1$$
$$m_\lambda=2(l+1) \textit{ if } k=2,\; m\geq 1$$
$$m_\lambda=2l+1 \textit{ if } \; k\geq 3, m=0$$
$$m_\lambda=\begin{cases}\frac{l}{2} & \textit{if } l \textit{ is even}\\ \frac{l-1}{2}+1 & \textit{if } l \textit{ is odd}\end{cases} \quad \textit{if } m=k=0$$
$$m_\lambda=\begin{cases}l+\frac{l}{2}+1 & \textit{if } l \textit{ is even}\\ l+\frac{l+1}{2}+1 & \textit{if } l \textit{ is odd}\end{cases} \quad \textit{if } k=2, m=0$$

If $\lambda=(k,3,2^m,1^l)$, *then*

$$m_\lambda=l+1.$$

Proof. If $\lambda=(k,2^m,1^l)$ and $k\geq 3, m\geq 1$, by the Littlewood-Richardson rule one has that λ occurs in $(k,2^m,1^{l-1})\otimes(1)$, $(k,2^{m-1},1^{l+1})\otimes(1)$, $(k-1,2^m,1^l)\otimes(1)$ and by Lemma 6.2, their multiplicities are l, $l+2$, $l+1$ respectively. Then the total multiplicity of $(k,2^m,1^l)$ is $l+l+2+l+1=3l+3=3(l+1)$.

If $\lambda=(2^m,1^l)$ and $m\geq 1$, by the Littlewood-Richardson rule one has that λ occurs in $(2^m,1^{l-1})\otimes(1)$, $(2^{m-1},1^{l+1})\otimes(1)$, and by Lemma 6.2, their multiplicities are l, and $l+2$ respectively. Then the total multiplicity of $(2^m,1^l)$ is $l+l+2=2(l+1)$.

If $\lambda=(k,1^l)$ one has that λ occurs in $(k,1^{l-1})\otimes(1)$, $(k-1,1^l)\otimes(1)$, and by Lemma 6.2, their multiplicities are l, and $l+1$ respectively. Then the total multiplicity of $(2^m,1^l)$ is $l+l+1=2l+1$.

If $k=m=0$, one has that if l is even, l comes from $(1^{l-1})\otimes(1)$, then $l-1$ is odd and by previous Lemma, its multiplicity is $\frac{l-1-1}{2}+1=\frac{l}{2}$; if l is odd, l comes from $(1^{l-1})\otimes(1)$, then $l-1$ is even and by previous Lemma, its multiplicity is $\frac{l-1}{2}+1$.

Finally, if If $\lambda=(k,3,2^m,1^l)$, one has that λ occurs in $(k,2^{m+1},1^l)\otimes(1)$ only and its multiplicity is $l+1$. The other cases are treated similarly. □

Lemma 6.4. $\frac{S_{(1)}-1}{\prod_{i=1}^m(1-x_i)}(H^B(E,\bar t))^2=\sum m_\lambda S_\lambda$, *where*

$$\lambda=(k,2^m,1^l)$$
$$\textit{or } \lambda=(k,3,2^m,1^l).$$

If $\lambda = (k, 2^m, 1^l)$, *then*

$$m_\lambda = 2(l+1) \text{ if } k \geq 3, m \geq 1$$
$$m_\lambda = l+1 \text{ if } k = 2,\ m \geq 2$$
$$m_\lambda = l \text{ if } k \geq 3, m = 0$$
$$m_\lambda = \begin{cases} -1 & \text{if } l \text{ is even} \\ 0 & \text{if } l \text{ is odd} \end{cases} \quad \text{if } m = k = 0$$
$$m_\lambda = \begin{cases} \frac{l}{2} & \text{if } l \text{ is even} \\ \frac{l+1}{2} & \text{if } l \text{ is odd} \end{cases} \quad \text{if } k = 2, m = 0$$

If $\lambda = (k, 3, 2^m, 1^l)$, *then*

$$m_\lambda = l + 1.$$

Proof. We have just to combine Lemmas 6.2 and 6.3. □

Now we have all the results to state the following proposition:

Proposition 6.1. $H^B(R, \bar{t}) = 2H^B(E, \bar{t}) + \frac{S_{(1)}-1}{\prod_{i=1}^m (1-x_i)} (H^B(E, \bar{t}))^2 = \sum m_\lambda S_\lambda$, *where*

$$\lambda = (k, 2^m, 1^l)$$
$$\text{or } \lambda = (k, 3, 2^m, 1^l).$$

If $\lambda = (k, 2^m, 1^l)$, *then*

$$m_\lambda = 2(l+1) \text{ if } k \geq 3, m \geq 1$$
$$m_\lambda = l+1 \text{ if } m \geq 2$$
$$m_\lambda = l \text{ if } k \geq 3, m = 0$$
$$m_\lambda = \begin{cases} 1 & \text{if } l \text{ is even} \\ 0 & \text{if } l \text{ is odd} \end{cases} \quad \text{if } m = k = 0$$
$$m_\lambda = \begin{cases} \frac{l}{2} & \text{if } l \text{ is even} \\ \frac{l+1}{2} & \text{if } l \text{ is odd} \end{cases} \quad \text{if } k = 2, m = 0$$

If $\lambda = (k, 3, 2^m, 1^l)$, *then*

$$m_\lambda = l + 1.$$

Proof. We have to add 2 at the multiplicity of (1^l) where l is even, computed in the previous lemma. □

Now, we have all informations about proper Hilbert series of the algebra R but, at the light of Proposition 3.3, we have that

$$H(R, \bar{t}) = \frac{1}{\prod_{i=1}^m 1 - t_i} H^B(R, \bar{t}),$$

so we can state the following proposition:

Proposition 6.2. $H(R, \bar{t}) = \sum m_\lambda S_\lambda$, *where*

$$\lambda = (k_1, k_2, 2^m, 1^l) \text{ or } \lambda = (k_1, k_2, 3, 2^m, 1^l).$$

If $\lambda = (k_1, k_2, 2^m, 1^l)$, *then*

$$m_\lambda = 12(k_1 - k_2 + 1)(l+1) \quad \text{if } k_1 \geq k_2 \geq 3,\ m \geq 1$$
$$m_\lambda = 4(k_1 - k_2 + 1)(2l+1) \quad \text{if } k_1 \geq k_2 \geq 3,\ m = 0$$
$$m_\lambda = 8(k_1 - 2)(l+1) + 4(l+1) \quad \text{if } k_1 \geq k_2 = 2,\ m \geq 1$$
$$m_\lambda = 3(k_1 - 2)(2l+1) + 3l + 2 \quad \text{if } k_1 \geq k_2 = 2,\ m = 0$$
$$m_\lambda = (k_1 - 2)(2l-1) + l + 1 \quad \text{if } k_1 \geq 2,\ k_2 = 0,\ m = 0,\ l \geq 1$$
$$m_\lambda = 1 \quad \text{if } \lambda = (1^l) \text{ or } \lambda = (k).$$

If $\lambda = (k_1, k_2, 2^m, 1^l)$, *then*

$$m_\lambda = 4(k_1 - k_2 + 1)(l + 1) \text{ if } k_2 \geq 3m \geq 1.$$

PROOF. Suppose firstly $k_2 \geq 3, m \geq 1$, then by Proposition 3.3 and the following Remark, we have that

$$m_\lambda = \sum_{i=0}^{k_1-k_2} m_{(k_1-i,3,2^m,1^l)} + m_{(k_1-i,3,2^{m-1},1^{l+1})} + m_{(k_1-i,3,2^m,1^{l-1})} + m_{(k_1-i,3,2^{m-1},1^l)} +$$

$$+ \sum_{i=0}^{k_1-k_2} m_{(k_1-i,2^{m+1},1^l)} + m_{(k_1-i,2^m,1^{l+1})} + m_{(k_1-i,2^{m+1},1^{l-1})} + m_{(k_1-i,2^m,1^l)}$$

and, by Proposition 6.1, $m_\lambda = (k_1 - k_2 + 1)(l + 1 + l + 2 + l + l + 1) + (k_1 - k_2 + 1)(2(l + 1) + 2(l + 2) + 2l + 2(l + 1)) = 4(k_1 - k_2 + 1)(l + 1) + 8(k_1 - k_2 + 1) = 12(k_1 - k_2 + 1)(l + 1)$.

Let $\lambda = (k)$, then we have that

$$m_\lambda = \sum_{i=0}^{k-1} m_{(k-i)} + m_1$$

and, by Proposition 6.1, $m_\lambda = 1$.

Finally, Let $\lambda = (1^l)$, then we have that

$$m_\lambda = m_{(1^l)} + m_{(1^{l-1})}$$

and, by Proposition 6.1, $m_\lambda = 1$. The other cases are treated similarly. □

Example 6.1. *Using Proposition* 3.2 *and the previous proposition, we have that*

$$\chi_1(R) = (1),$$

$$\chi_2(R) = (2) + (1^2),$$

$$\chi_3(R) = (3) + 2(2, 1) + (1^3),$$

$$\chi_4(R) = (4) + 3(3, 1) + 2(2^2) + 3(2, 1^2) + (1^4),$$

$$\chi_5(R) = (5) + 4(4, 1) + 5(3, 2) + 6(3, 1^2) + 5(2^2, 1) + 4(2, 1^3) + (1^5),$$

$$\chi_6(R) = (6) + 5(5, 1) + 8(4, 2) + 9(4, 1^2) + 14(3, 2, 1) + 4(3^2) + 9(3, 1^3) + (2^3) + 8(2^2, 1^2) + 5(2, 1^4) + (1^6).$$

7. $\mathbb{Z}_2$-GRADED COCHARACTERS OF $UT_2(E)$

We compute directly the $\mathbb{Z}_2$-graded cocharacters of $UT_2(E)$ starting from the cocharacters of $UT_2(F)$, then we will use a formula due to Di Vincenzo and Nardozza (see [4]).

Using direct computations, is easy to prove (indeed, see [2]) that the ordinary cocharacters sequence of

$$S = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$$

is the following

Proposition 7.1. *For any* $n \geq 0$, $\chi_n(S) = \sum_{i=1}^{3} m_\lambda^{(i)} S_{\lambda^{(i)}}$, *where*

$$\lambda^{(1)} = (n)$$
$$\lambda^{(2)} = (k_1, k_2)$$
$$\lambda^{(3)} = (k_1, k_2, 1)$$

and

$$\begin{aligned} m_{\lambda^{(1)}} &= 1 \\ m_{\lambda^{(2)}} &= k_1 - k_2 + 1 \\ m_{\lambda^{(3)}} &= k_1 - k_2 + 1. \end{aligned}$$

The algebra R has a natural structure of $\mathbb{Z}_2$-graded algebra, where

$$R^0 = \begin{pmatrix} E^0 & E^0 \\ 0 & E^0 \end{pmatrix}$$

and

$$R^1 = \begin{pmatrix} E^1 & E^1 \\ 0 & E^1 \end{pmatrix},$$

and E^0, E^1 are, respectively, the 0 and the 1 part of the natural $\mathbb{Z}_2$-grading of E. As a $\mathbb{Z}_2$-graded algebra, R is naturally isomorphic to $S \otimes E$. We have the following proposition due to Di Vincenzo and Nardozza:

Proposition 7.2. *Let $k, l \in \mathbb{N}$ such that $k + l = n$ and consider $H = S_k \times S_l$. If $(\chi_n(S))_{\downarrow H} = \sum m_{\lambda,\mu} \lambda \otimes \mu$, then*

$$\chi_n^{\mathbb{Z}_2}(R) = \sum_{k+l=n} \sum_{\substack{\lambda \vdash k \\ \mu \vdash l}} m_{\lambda,\mu} \lambda \otimes \mu'.$$

At the light of Proposition 7.2, it suffices to know $(\chi_n(S))_{\downarrow H} = \sum m_{\lambda,\mu} \lambda \otimes \mu$, then we have immediately the n-th $\mathbb{Z}_2$-graded cocharacter of $UT_2(E)$. At this purpose we have the following Proposition.

Proposition 7.3. *For any $n \geq 0$, $(\chi_n(S))_{\downarrow H} = \sum m_{\lambda,\mu} \lambda \otimes \mu$, where*

$$\begin{aligned} \lambda &= (\lambda_1, \lambda_2, \lambda_3), \quad \lambda_3 \leq 1 \\ \mu &= (\mu_1, \mu_2, \mu_3), \quad \mu_3 \leq 1. \end{aligned}$$

More precisely,

$$(n)_{\downarrow H} = (k) \otimes (l)$$

$$(k_1, k_2)_{\downarrow H} = \begin{cases} (k) \otimes (l) \\ (\lambda_1, \lambda_2) \otimes (l) \\ (\lambda_1, \lambda_2) \otimes (\mu_1, \mu_2) \end{cases}$$

$$(k_1, k_2, 1)_{\downarrow H} = \begin{cases} (\lambda_1, \lambda_2) \otimes (l) \\ 2((\lambda_1, \lambda_2) \otimes (\mu_1, \mu_2)) & \textit{if } \mu_1 - 1 \geq \mu_2 \\ (\lambda_1, \lambda_2) \otimes (\mu_1, \mu_2) & \textit{if } \mu_1 - 1 < \mu_2 \\ (\lambda_1, \lambda_2, 1) \otimes (l) \\ (\lambda_1, \lambda_2, 1) \otimes (\mu_1, \mu_2) \end{cases}$$

Proof. The result follows from a straightforward computation using the decomposition of $\chi_n(S)$ given in Proposition 7.1. More precisely, it follows by Branching Rule (see [12], Chapter 2) that when we restrict the representation $\nu = \sum(k_1, k_2, k_3)$ of S_n, with $k_3 \leq 1$, to its subgroup H, then its H-irreducible components are $\lambda \otimes \mu$, where $\lambda = (k_1', k_2', k_3')$ and $\mu = (l_1, l_2, l_3)$ are such that ν appears in the tensor product $(\lambda \otimes \mu)^{\uparrow S_n}$. By Frobenius Multiplicity Law (see [12]), the multiplicity of $\lambda \otimes \mu$ in the previous decomposition equals the multiplicity of ν in the induced representation $(\lambda \otimes \mu)^{\uparrow S_n}$. We will argue for the irreducible cocharacters of $\chi_n(F)$, i.e. (n), (k_1, k_2) where $k_2 \geq 1$ and $(k_1, k_2, 1)$. In the first case, it

is easy to see that $(n)_{\downarrow H} = m_{\lambda,\mu}((l)\otimes(k))$, where $\lambda = (k)$ and $\mu = (l)$. By the Littlewood-Richardson Rule, the multiplicity of (n) in the induced representation $(\lambda\otimes\mu)^{\uparrow S_n}$ is 1. Let $\nu = (k_1, k_2)$. Then

$$(k_1,k_2)_{\downarrow H} = \begin{cases} m_{\lambda,\mu}((k)\otimes(l)) & a) \\ m_{\lambda,\mu}((\lambda_1,\lambda_2)\otimes(l)) & b) \\ m_{\lambda,\mu}((\lambda_1,\lambda_2)\otimes(\mu_1,\mu_2)) & c) \end{cases}$$

The case $a)$ has been yet treated. Consider the case $b)$. Here, the only way to obtain ν is adding $k_2-\lambda_2$ $\boxed{1}$ to λ_2 so the multiplicity of ν in $(\lambda_1,\lambda_2)\otimes(l)$ is 1. Even in the case $c)$, the only possible way to obtain ν is adding all the boxes $\boxed{2}$ to λ_2 so the multiplicity of ν in $(\lambda_1,\lambda_2)\otimes(l)$ is still 1. Finally, let $\nu = (k_1,k_2,1)$. Then

$$(k_1,k_2,1)_{\downarrow H} = \begin{cases} m_{\lambda,\mu}((\lambda_1,\lambda_2)\otimes(l)) & a) \\ m_{\lambda,\mu}((\lambda_1,\lambda_2)\otimes(\mu_1,\mu_2)) & b) \\ m_{\lambda,\mu}((\lambda_1,\lambda_2,1)\otimes(l)) & c) \\ m_{\lambda,\mu}((\lambda_1,\lambda_2,1)\otimes(\mu_1,\mu_2)) & d) \end{cases}$$

Firstly, we note that the cases $a), c), d)$ are similar to those computed for $(k_1,k_2)_{\downarrow H}$. Thus we have to argue only for the case $b)$. Suppose $\mu_1 - 1 \geq \mu_2$, then we have to add a final box $\boxed{1}$ or $\boxed{2}$ to λ if $\mu_1 - 1 \geq \mu_2$, finally we have to add all the remaining $\boxed{2}$ in the only possible way. If $\mu_1 - 1 < \mu_2$, we have to add only the final box $\boxed{2}$ to λ, finally we have to add all the remaining $\boxed{2}$ in the only possible way, from which the assertion.

□

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